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THE CONSTRUCTION OF STABLE BRIDGES IN DIFFERENTIAL GAMES WITH PHASE CONSTRAINTS†

V. Yu. PAKHOTINSKIKH, A. A. USPENSKII and V. N. USHAKOV

Ekaterinburg

e-mail: ushak@imm.uran.ru, uspen@imm.uran.ru, bigvasily@mail.ru

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A differential approach-and-evasion game in a finite time interval is considered [1]. It is assumed that the positions of the game are constricted by certain constraints which represent a closed set in the space of the positions. In the case of the first player, it is necessary to ensure that the phase point falls into the terminal set at a finite instant of time and, in the case of the second player, that this terminal set is evaded at this instant [1]. A method is proposed for the approximate construction of the positional absorption set, that is, the set of all positions belonging to a constraint from which the problem of approach facing the first player is solvable. Relations are written out which determine the system of sets which approximates the positional absorption set. The main result is a proof of the convergence of the approximate system of sets to the positional absorption set and the construction of a computational procedure for constructing the approximate system of sets. © 2003 Elsevier Ltd. All rights reserved.

This paper touches on the earlier investigations in [1–16].

1. FORMATION OF THE PROBLEM

Suppose a conflict-control problem is specified, the behaviour of which is described by the equation

$$\dot{x} = f(t, x, u, \upsilon), \quad x(t_0) = x_0, \quad u \in P, \quad \upsilon \in Q \tag{1.1}$$

in the time interval $[t_0, \vartheta]$ $(t_0 \le \vartheta < \infty)$.

Here, x is an m-dimensional phase vector from the Euclidean space R^m, u is the control of the first player, υ is the control of the second player, and P and Q are compacta in the Euclidean spaces R^p and R^q respectively.

It is assumed that the following conditions are satisfied:

A. The game takes place in a bounded and closed domain Φ of the space of the variables t, $x(t \in [t_0, \vartheta], x \in R^m)$.

B. The vector function $f(t, x, u, \upsilon)$ is defined and is continuous with respect to the union (t, x, u, υ) in the set $I \times R^m \times P \times Q$ (I is the time interval containing $[t_0, \vartheta]$ within it) and satisfies a local Lipschitz condition with respect to x: for any compactum $D \subset [t_0, \vartheta] \times R^m$, and $L = L(D) \in (0, \infty)$ is found such that

$$\|f(t, x^{(1)}, u, \upsilon) - f(t, x^{(2)}, u, \upsilon)\| \le L\|x^{(1)} - x^{(2)}\| \tag{1.2}$$

for any $(t, x^{(i)}, u, \upsilon)$ $(i = 1, 2)$ from $D \times P \times Q$.

Here, $\|f\|$ is the norm of the vector f in the corresponding Euclidean space.

It is also assumed that the motions $x(t)$ of system (1.1) are extendable in the interval $[t_0, \vartheta]$.

The differential game being considered is made up an approach problem and an evasion problem [1]. In the problem of approach facing the first player, it is necessary to ensure that the motions $x(t)$ of system (1.1) at the instant of time ϑ fall in the closed set M, which is contained in $\Phi(\vartheta) = \{x \in R^m : (\vartheta, x) \in \Phi\}$. It is necessary to ensure a solution of the problem in the class of positional procedures for the control of the first player [1].

It has been shown [1] that an alternative holds in the case of the differential game which has been formulated and this is that a closed set $W^0 \subset \Phi$ exists, which is called the positional absorption set, such that the approach problem is solvable for all initial positions $(t_*, x_*) \in W^0$ and the evasion problem is solvable for all initial positions $(t_*, x_*) \in \Phi \backslash W^0$. At the same time, it was established that W^0 is the maximum u-stable bridge.

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The set W^0 only admits of an analytical description in infrequent cases, and the problem of the approximate construction of the set W^0, considered below, is therefore important. Using retrograde constructions, a system of sets is introduced which approximates W^0 and corresponds to a certain discretization of the interval $[t_0, \vartheta]$. The convergence of the approximating system of sets to W^0 is validated in the case of a discretization step tending to zero.

2. THE STABLE ABSORPTION OPERATOR AND STABLE BRIDGES

Since W^0 is the maximum u-stable bridge [1], the property of stability is the key property in separating out W^0 in Φ. Stability of the set W, which is contained in Φ, signifies a weak invariance of W with respect to a certain family of differential inclusions related to system (1.1), which are also considered in the interval $[t_0, \vartheta]$.

We will now introduce a function (the Hamiltonian) of system (1.1) into the treatment.

$$H(t, x, l) = \max_{u \in P} \min_{v \in Q} \langle l, f(t, x, u, v) \rangle, \quad l \in R^m$$

where $\langle l, f \rangle$ is the scalar product of the vectors l and f from R^m.

Suppose $\Phi_\alpha(\sigma > 0)$ is the σ-neighbourhood of the set Φ in the space of the variables t and x such that $\Phi_\alpha \subset I \times R^m$.

We will assume that $G = \{f \in R^m : \|f\| \le K < \infty\}$ is a sphere in R^m such that

$$F(t, x) = \operatorname{co}\{f(t, x, u, v) : u \in P,\ v \in Q\} \subset G, \quad \forall (t, x) \in \Phi_\sigma$$

Here, $\operatorname{co}\{f\}$ is the convex shell of the set f.

Suppose a certain set Ψ of elements ψ and also the family $\{F_\psi : \psi \in \Psi\}$ of mappings $F_\psi : (t, x) \mapsto F_\psi(t, x)$, $(t, x) \in \Phi_\sigma$, which satisfies the following conditions, are given.

A.1. For any $(t, x, \psi) \in \Phi_\sigma \times \Psi$, the set $F_\psi(t, x)$ is convex, closed in R^m and $F_\psi(t, x) \subset G$.

A.2. The inequality

$$\min_{\psi \in \Psi} h_{F_\psi(t, x)}(l) = H(t, x, l)$$

holds for any $(t, x, l) \in \Phi_\sigma \times S$.

A.3. A function $\omega^*(\delta)$ ($\omega^*(\delta) \downarrow 0$ exists when $\delta \downarrow 0$) such that

$$d(F_\psi(t_*, x_*), F_\psi(t^*, x^*)) \le \omega^*(|t_* - t^*| + \|x_* - x^*\|)$$
$$(t_*, x_*) \text{ and } (t^*, x^*) \text{ from } \Phi_\sigma, \psi \in \Psi$$

Here

$$h_F(l) = \sup_{f \in F} \langle l, f \rangle \text{ when } F \subset R^m; \quad S = \{l \in R^m : \|l\| = 1\}$$

and $d(F_*, F^*)$ is the Hausdorff distance between the sets F_* and F^* in R^m.

As examples of families of mappings which satisfy conditions A.1 – A.3, we mention the families $\{G_l : l \in S\}$ and $\{F_{\upsilon(\cdot)} : \upsilon(\cdot) \in V\}$ [1, 4, 5]; here, $G_l(t, x) = \{f \in G : \langle l, f \rangle \le H(t, x, l)\}$, $F_{\upsilon(\cdot)}(t, x) = \operatorname{co}\{f(t, x, u, \upsilon(u)) : u \in P\}$, $\operatorname{co}\{f\}$ is the closed convex shell if the set $\{f\}$ and V is the union of all of the mappings $\upsilon(\cdot) : P \mapsto Q$.

Note that, in the case of certain classes of control systems and, in particular, for systems (1.1) with right-hand side

$$f(t, x, u, v) = \phi(t, x) + B(t, x)u + C(t, x)v \tag{2.1}$$

and constraints P and Q, which are convex polyhedra with a finite number of vertices, a family of mappings can be introduced which satisfies conditions A.1–A.3 and corresponds to the finite set Ψ. The set of all the vertices $\upsilon^{(j)} (j = 1, 2, \ldots, J)$ of the polyhedron Q can be taken as this set Ψ and the set

$$F_{\upsilon^{(j)}}(t, x) = \{f = \phi(t, x) + B(t, x)u + C(t, x)\upsilon^{(j)} : u \in P\}, \quad j = 1, 2, \ldots, J$$

can be taken as the set $F_\psi(t, x)$.

This specification of the family $\{F_{\upsilon^{(j)}}, j = 1, 2, \dots, J\}$, which satisfies conditions A.1–A.3, enables one effectively to carry out an approximate construction of the set W^0.

We will now present a definition of the stable absorption operator in the approach problem being considered.

We introduce the following notation: $X_\psi(t^*; t_*, x_*)$ is the set of all $x^* \in R^m$ into which the solutions $x(\cdot) = (x(t): t_* \le t \le t^*)$ of the differential inclusion

$$\dot{x} \in F_\psi(t, x), \quad x(t_*) = x_*,$$

$$X_\psi^{-1}(t_*; t^*, X^*) = \{x_* \in R^m : X_\psi(t^*; t_*, x_*) \cap X^* \ne \varnothing\}$$

enter at the instant of time $t^* \in [t_*, \vartheta]$ and X^* is a set from R^m.

Definition 1. In the approach problem, we shall call the mapping

$$(t_*, t^*, X^*) \mapsto 2^{R^m} (t_*, t^*, X^*) \in \Delta \times 2^{R^m}$$

defined by the relation

$$\pi(t_*, t^*, X^*) = \Phi(t_*) \cap \Big(\bigcap_{\psi \in \Psi} X_\psi^{-1}(t_*; t^*, X^*) \Big)$$

the stable absorption operator π. Here, $\Delta = \{(t_*, t^*): t_0 \le t_* < t^* \le \vartheta\}$.

Definition 2. We shall call the closed set $W \subset \Phi$ a u-stable bridge in the approach problem, if

$$W(\vartheta) \subset M, \; W(t_*) \subset \pi(t_*; t^*, W(t^*)), \; \forall (t_*, t^*) \in \Delta$$

Here, $W(t) = \{x \in R^m : (t, x) \in W\}$.

The problem of stability is central in the theory of positional differential games. Definition 2 presented here is a more recent definition of stability. However, it can be shown that the u-stable bridges W^0, which are separated out in the set Φ using the definition from [1], are exactly the same bridges. This justifies the use of the family $\{F_\psi : \psi \in \Psi\}$, which satisfies conditions A.1–A.3, is separating out the maximum u-stable bridge W^0, that is, the positional absorption set [1], in Φ. Note that, in solving the approach problem using the positional approach, the main difficulty lies in constructing the bridge W^0. It is well known that an accurate description of the bridge W^0 using analytical relations is only possible in the case of certain special classes of systems (1.1). In the general case, the exact separation of the stable bridge W^0 has to be abandoned and, because of this, the problem of the approximate construction of W^0 is an urgent one. The following section is concerned with this problem.

3. THE APPROXIMATING SYSTEM OF SETS

We will assume that the family $\{F_\psi : \psi \in \Psi\}$ also satisfies the following condition.

A.4. A number $\lambda \in (0, \infty)$ exist such that

$$d(F_\psi(t, x_*), F_\psi(t, x^*)) \le \lambda \| x_* - x^* \|$$

for any $\psi \in \Psi$, (t, x_*) and (t, x^*) from Φ_σ.

We will now present a definition of the approximating system of sets which is directed towards the approximate calculation of the set W^0. The concept of an approximating system of sets arises when a scheme which is continuous (with respect to time) is replaced by a u-stable discrete scheme, where the interval $[t_0, \vartheta]$ is replaced by the subdivision $\Gamma = \{t_0, t_1, \dots, t_n = \vartheta\}$ and the sets $X_\psi(t^*; t_*, x_*)$, $\psi \in \Psi$ from Definition 1 are replaced by the sets $x_* + (t^* - t_*) F_\psi(t_*, x_*)$, $\psi \in \Psi$. Furthermore, Definitions 1 and 2 are transformed in a corresponding manner into definitions which are intended for work with a discrete time t_i $(i = 0, 1, \dots, N)$.

So, we put

$$\tilde{X}_\psi(t^*; t_*, x_*) = x_* + (t^* - t_*)F_\psi(t_*, x_*)$$

$$\tilde{X}_\psi^{-1}(t_*; t^*, X^*) = \{x_* \in R^m : X^* \cap \tilde{X}_\psi(t_*; t_*, x_*) \neq \varnothing\}$$

$$(t_*, t^*) \in \Delta, \quad x_* \in R^m, X^* \subset R^m, \quad \psi \in \Psi$$

Definition 3. We shall call the mapping $(t_*, t^*, X^*) \mapsto \Delta \times 2^{R^m}$, given by the relation

$$\pi^\varepsilon(t_*, t^*, X^*) = \Phi(t_*)_\varepsilon \cap \Big(\bigcap_{\psi \in \Psi} \tilde{X}_\psi^{-1}(t_*; t^*, X^*)\Big)$$

the approximating stable absorption operator $\pi^\varepsilon(\varepsilon \in (0, \sigma))$ in the approach problem.

We will use the notation

$$\begin{aligned} \omega^*(\delta) &= \sup_{\substack{(t_*, x_*) \text{ and } (t^*, x^*) \text{ from } \Phi_\sigma \\ |t_* - t^*| + \|x_* - x^*\| \le \delta}} \omega^*(|t_* - t^*| + \|x_* - x^*\|) \\ \omega(\delta) &= \delta\omega^*((1 + K)\delta), \quad \delta > 0 \end{aligned} \tag{3.1}$$

It follows from the definition of the functions $\omega^*(\delta)$ and $\omega(\delta)$ that they decay monotonically to zero when $\delta \downarrow 0$ and $\lim \dfrac{\omega^*(\delta)}{\delta} = 0$.

We specify the sequence of subdivisions $\Gamma_n = \{t_0, t_1, \dots, t_{N(n)} = \vartheta\}$ of the interval $[t_0, \vartheta]$ such that the diameters

$$\Delta^{(n)} = \max\{\Delta_i : 0 \le i \le N(n) - 1\}, \quad \Delta_i = t_{i+1} - t_i$$

of the subdivisions Γ_n tend monotonically to zero as $n \to \infty$.

Note that the instants t_i of the subdivisions Γ_n are different for each subdivision Γ_n. However, in order not to make the notation too complicated, we shall not explicitly reflect this dependence of the instants t_i on the number n.

We will not establish a correspondence between a sequence of numbers $\{\varepsilon_i\}$

$$\varepsilon_i = \omega(\Delta_{i-1}) + (1 + \lambda\Delta_{i-1})\varepsilon_{i-1}, \quad i = 1, 2, \dots, N(n), \quad \varepsilon_0 = 0$$

and each subdivision Γ_n.

We shall also assume that the subdivisions Γ_n are chosen to be so "minute" that, for any Γ_n, the inequalities

$$\max_{0 \le i \le N(n) - 1} (1 + K)\Delta_i = (1 + K)\Delta^{(n)} \le \sigma, \qquad \max_{0 \le i \le N(n) - 1} \varepsilon_i \le \sigma \tag{3.2}$$

are satisfied.

Corresponding to each subdivision, we now set up a sequence $\{\tilde{W}^{(n)}(t_i)\}$ of sets $\tilde{W}^{(n)}(t_i) \subset R^m$, $t_i \in \Gamma_n$ which is given by recurrence relations starting from the final instant $t_{N(n)} = \vartheta$ of the subdivision Γ_n.

Definition 4. Assume that

$$\tilde{W}^{(n)}(\vartheta) = M_{\varepsilon_{N(n)}}$$

$$\tilde{W}^{(n)}(t_i) = \pi^{\varepsilon_i}(t_i; t_{i+1}, \tilde{W}^{(n)}(t_{i+1})), \quad i = N(n) - 1, \quad N(n) - 2, \dots, 1, 0$$

The sequence $\{\tilde{W}^{(n)}(t_i)\}$ is therefore a sequence of the sets $\tilde{W}^{(n)}(t_i) \subset R^m$ given in a backwards manner. We will now determine the limit of this sequence when the diameter $\Delta^{(n)}$ of a subdivision Γ_n tends to zero.

Definition 5. We will assume that Ω^0 is the set of all points $(t_*, x_*) \in \Phi$ for each of which a sequence

$$\{(\tau_n, x_n) : \tau_n = t_n(t_*),\ x_n \in \tilde{W}^{(n)}(\tau_n)\} \tag{3.3}$$

is found such that $(t_*, x_*) = \lim(\tau_n, x_n)$ when $n \mapsto \infty$. Here

$$\tau_n(t_*) = \min_{t_i \in \Gamma_n,\ t_i \geq t_*} t_i$$

Since the equality $\tilde{W}^{(n)}(\vartheta) = M_{\varepsilon_{N(n)}}$ is satisfied, the section $\Omega^0(\vartheta) = \{x \in R^m : (\vartheta, x) \in \Omega^0\}$ of the set Ω^0 is defined by the equality $\Omega^0(\vartheta) = M$. This means that $\Omega^0 \neq \emptyset$. Moreover, it follows from Definition 5 that $\Omega^0 \subset \Phi$.

Theorem. The equality

$$\Omega^0 = W^0$$

holds.

Proof. We will first prove the inclusion $\Omega^0 \subset W^0$. In order to do this, we will show that Ω^0 is a u-stable bridge. Actually, the inclusion $\Omega^0(\vartheta) \subset M$, which follows from the equality $\Omega^0(\vartheta) = M$, is satisfied.

We will now prove the inclusion $\Omega^0(t_*) \subset \pi(t_*; t^*, \Omega^0(t^*))$ for any $(t_*, t^*) \in \Delta$.

We fix an arbitrary point $(t_*, x_*) \in \Omega^0$, $t_* < \vartheta$ for this and use Definition 5.

We consider an arbitrary number n and the interval $[\tau_n, \vartheta]$ corresponding to it. It follows from the inclusion $x_n \in \tilde{W}^{(n)}(\tau_n)$ that a vector function $\tilde{x}^{(n)}(t)$, which is absolutely continuous in $[\tau_n, \vartheta]$, exists such that the relations

$$\begin{aligned} &\dot{\tilde{x}}^{(n)}(t) \in F_\psi(t_i, \tilde{x}^{(n)}(t_i)), \quad t \in [t_i, t_{i+1}] \subset [\tau_n, \vartheta) \\ &\tilde{x}^{(n)}(\tau_n) = x_n, \quad \tilde{x}^{(n)}(t_i) \in \tilde{W}^{(n)}(t_i), \quad \tau_n < t_i < \vartheta \end{aligned} \tag{3.4}$$

hold for any $\psi \in \Psi$.

We now introduce functions into the treatment which are continuous extensions of the functions $\tilde{x}^{(n)}(t)$, $t \in [\tau_n, \vartheta]$ in the interval $[t_*, \vartheta]$.

$$\tilde{y}^{(n)}(t) = \begin{cases} \tilde{x}^{(n)}(\tau_n), & t_* \leq t \leq \tau_n \\ \tilde{x}^{(n)}(t), & \tau_n \leq t \leq \vartheta \end{cases}, \quad n = 1, 2, \ldots$$

Since the sequence $\{\tilde{y}^{(n)}(t)\}$ is uniformly bounded and equipotentially continuous in $[t_*, \vartheta]$, a uniformly converging subsequence can be separated out from it. Without loss of generality in the reasoning, we shall assume that this same sequence $\{\tilde{y}^{(n)}(t)\}$ converges uniformly in $[t_*, \vartheta]$. On putting $x(t) = \lim \tilde{y}^{(n)}(t)$, $t \in [t_*, \vartheta]$ (the limit is henceforth chosen when $n \to \infty$), we obtain

$$\begin{aligned} &x(t_*) = \lim \tilde{y}^{(n)}(t_*) = \lim \tilde{x}^{(n)}(\tau_n) = \lim x_n = x_* \\ &x(t) = \lim \tilde{y}^{(n)}(t) = \lim \tilde{x}^{(n)}(t), \quad t \in (t_*, \vartheta] \end{aligned} \tag{3.5}$$

It follows from (3.3)–(3.5) that the vector function $x(t)$, $t \in [t_*, \vartheta]$ satisfies the differential inclusion

$$\dot{x} \in F_\psi(t, x) \quad \text{almost everywhere in} \quad [t_*, \vartheta] \tag{3.6}$$

and the inclusion

$$(t, x(t)) \in \Omega^0, \quad t \in [t_*, \vartheta] \tag{3.7}$$

The inclusion (3.6) is proved in the standard manner (see [15], for example).

We will now prove the inclusion (3.7). We fix an arbitrary instant of time $t \in [t_*, \vartheta]$. The equality $x(t) = \lim \tilde{y}^{(n)}(t)$ holds for this instant. By the construction of the function $\tilde{y}^{(n)}(t)$, $t \in [t_*, \vartheta]$, the inclusion

$$\tilde{y}^{(n)}(t_n(t)) = \tilde{x}^{(n)}(t_n(t)) \in \tilde{W}^{(n)}(t_n(t))$$

is satisfied, where the instant of time is defined above.

We put

$$\eta_n = t_n(t) \text{ and } y_n = \tilde{x}^{(n)}(\eta_n) = \tilde{x}^{(n)}(t_n(t))$$

Then

$$\|(t, x(t)) - (\eta_n, y_n)\| \le \|(t, x(t)) - (t, \tilde{y}^{(n)}(t))\| +$$
$$+ \|(t, \tilde{y}^{(n)}(t)) - (t_n(t), \tilde{y}^{(n)}(t_n(t)))\| \le \|x(t) - \tilde{y}^{(n)}(t)\| + (1+K)\Delta^{(n)}$$

Taking into account this inequality and the limiting relations

$$x(t) = \lim \tilde{y}^{(n)}(t), \quad \lim \Delta^{(n)} = 0$$

we obtain

$$(t, x(t)) = \lim(\eta_n, y_n), \quad \eta_n = t_n(t), \quad y_n \in \tilde{W}^{(n)}(\eta_n)$$

The inclusion (3.7) is thereby proved.

So, a solution $x(t)$, $t \in [t_*, \vartheta]$ of the differential inclusion (3.6) is found, for any point $(t_*, x_*) \in \Omega^0$, $t_* < \vartheta$ and any $\psi \in \Psi$, which satisfies the inclusion (3.7). It follows from inclusions (3.6) and (3.7) that $\Omega^0(t_*) \in \pi\,(t_*; t^*, \Omega^0(t^*))$ for any $(t_*, t^*) \in \Delta$. This means that Ω^0 is a u-stable bridge in the approach problem being considered, and that $\Omega^0 \subset W^0$.

We will now prove the inverse inclusion $W^0 \subset \Omega^0$.

We consider a subdivision Γ_n of the interval $[t_0, \vartheta]$ and all the non-empty sections $W^0(t_i)$, $t_i \in \Gamma_n$ of bridge W^0. We use the notation

$$T_n = \{t_i \in \Gamma_n : W^0(t_i) \ne \varnothing\}$$

The set T_n is non-empty. Since $W^0(t_N) = M \ne \varnothing$. Moreover, the set T_n possesses the following property: if $t_i \in T_n$ then $t_{i+1} \in T_n$.

According to Definitions 1 and 2, the inclusions

$$W^0(t_i) \subset \Phi(t_i) \cap X_\psi^{-1}(t_i; t_{i+1}, W^0(t_{i+1})), \quad t_i \in T_n, \quad \forall \psi \in \Psi \tag{3.8}$$

hold.

We select an arbitrary instant of time $t_i \in T_n$, $t_i < \vartheta$ and consider the sets $W^0(t_i)$ and $W^0(t_{i+1})_{\omega(\Delta_i)}$; the numbers $\omega(\Delta_i)$ are defined above.

The inclusion

$$W^0(t_i) \subset \tilde{X}_\psi^{-1}(t_i, t_{i+1}, W^0(t_{i+1})_{\omega(\Delta_i)}), \quad t_i \in T_n \tag{3.9}$$

holds.

We shall prove this. Suppose $x(t_i) \in W^0(t_i)$.

We consider an arbitrary solution $x(t)$, $t \ge t_i$ of the differential inclusion

$$\dot{x} \in F_\psi(t, x), \quad t \ge t_i$$

with an initial value $x(t_i)$.

Since $(t_i, x(t_i)) \in W^0 \subset \operatorname{int}\Phi_\sigma$, then, for all $t \in [t_i, t_{i+1}]$ sufficiently close to t_i, the inclusion $(t, x(t)) \in \operatorname{int}\Phi_\sigma$ holds. We will show that the inclusion $(t, x(t)) \in \operatorname{int}\Phi_\sigma$ holds for all $t \in [t_i, t_{i+1}]$ subject to special assumptions regarding Γ_n.

Let us assume that the opposite is true: there is an instant of time $t^\circ \in [t_i, t_{i+1}]$ at which the point $(t, x(t))$ reaches the boundary $\partial\Phi_\sigma$ of the set Φ_σ. Without loss in generality, we can assume that t° is the time when the point $(t, x(t))$, $t \geq t_i$ first reaches the boundary $\partial\Phi_\sigma$, that is

$$(t, x(t)) \in \text{int}\Phi_\sigma \quad \text{when} \quad t \in [t_i, t^\circ) \quad \text{and} \quad (t^\circ, x(t^\circ)) \in \partial\Phi_\sigma$$

It follows from this that the equality $\dot{x}(t) = f(t)$, where $\|f(t)\| \leq K$ is satisfied almost everywhere in $[t_i, t^\circ]$. The point $(t^\circ, x(t^\circ)$ then satisfies the inequality

$$\|(t^\circ, x(t^\circ)) - (t_i, x(t_i))\| \leq (t^\circ - t_i) + K(t^\circ - t_i) < (1 + K)\Delta_i < \sigma$$

Hence, the inclusion $(t^\circ, x(t^\circ)) \in \text{int}\Phi_\sigma$ also follows from the inclusion $(t_i, x(t_i)) \in \Phi$, which contradicts the definition of the instant t°. At the same time, it has been established that $(t, x(t)) \in \text{int}\Phi_\sigma \subset \Phi_\sigma$ for all $t \in [t_i, t_{i+1}]$.

We now consider the set $X_\psi(t_{i+1}; t_i, x(t_i))$ for the selected point $x(t_i) \in W^0(t_i)$. Each point $x(t_{i+1}) \in X_\psi(t_{i+1}; t_i, x(t_i))$ is the value at the instant of time t_{i+1} of a certain solution $x(t)$, $t \in [t_i, t_{i+1}]$ of the differential inclusion $\dot{x} \in F_\psi(t, x)$, $t \in [t_i, t_{i+1}]$ with an initial value $x(t_i)$. The equality

$$x(t_{i+1}) = x(t_i) + \int_{t_i}^{t_{i+1}} f(t)dt, \quad t \in [t_i, t_{i+1}]$$

holds where $f(t)$ is a Lebesque integrable function which satisfies the inclusion $f(t) \in F_\psi(t, x(t))$ almost everywhere in $[t_i, t_{i+1}]$.

Taking account of the inclusion

$$(t_i, x(t_i)) \in \Phi_\sigma, \quad (t, x(t)) \in \Phi_\sigma, \quad t \in [t_i, t_{i+1}]$$

and the definition of the function $\omega^*(\Delta)$, we obtain

$$d(F_\psi(t, x(t)), F_\psi(t_i, x(t_i))) \leq \omega^*(|t - t_i| + \|x(t) - x(t_i)\|) \leq \omega^*((1 + J)\Delta_i), \ t \in [t_i, t_{i+1}] \tag{3.10}$$

This means that the inclusion

$$f(t) \in F_\psi(t_i, x(t_i))_{\omega^*((1+K)\Delta_i)}$$

holds from which the inclusion

$$\frac{1}{\Delta_i}\int_{t_i}^{t_{i+1}} f(t)dt \in F_\psi(t_i; x(t_i))_{\omega^*((1+K)\Delta_i)} \tag{3.11}$$

follows.

The inclusion

$$x(t_{i+1}) \in \tilde{X}_\psi(t_{i+1}; t_i, x(t_i))_{\omega(\Delta_i)} \tag{3.12}$$

follows from (3.11).

Taking account of the fact that relation (3.12) was obtained for an arbitrary point $x(t_{i+1}) \in X_\psi(t_{i+1}; t_i, x(t_i))$, we conclude that

$$X_\psi(t_{i+1}; t_i, x(t_i)) \subset \tilde{X}_\psi(t_{i+1}; t_i, x(t_i))_{\omega(\Delta_i)}$$

It follows from the inclusion $x(t_i) \in W^0(t_i)$ that

$$W^0(t_{i+1}) \cap X_\psi(t_{i+1}; t_i, x(t_i)) \neq \varnothing$$

and this means that

$$W^0(t_{i+1})_{\omega(\Delta_i)} \cap \tilde{X}_\psi(t_{i+1}; t_i, x(t_i)) \neq \varnothing \tag{3.13}$$

Since the instant of time $t_i \in T_n$ and the point $x(t_i) \in W^0(t_i)$ are chosen arbitrarily, the inclusion (3.9) follows from (3.13).

Next, we define the system $\{\hat{W}^{(n)}(t_i) : t_i \in T_n\}$ of sets $\hat{W}^{(n)}(t_i)$ by the equalities $\hat{W}^{(n)}(t_i) = W^0(t_i)_{\varepsilon_i}$ (the numbers ε_i are defined above in Section 2). According to the definition of the sets $\hat{W}^{(n)}(t_i)$, $t_i \in T_n$, the inclusions $W^0(t_i) \subset \hat{W}^{(n)}(t_i)$, $t_i \in T_n$ are satisfied.

The inclusions

$$\hat{W}^{(n)}(t_i) \subset \tilde{X}_\psi^{-1}(t_i, t_{i+1}, \hat{W}^{(n)}(t_{i+1})), \quad t_i \in T_n \tag{3.14}$$

hold for any $\psi \in \Psi$.

We will now prove this. Suppose $x(t_i) \in \hat{W}^{(n)}(t_i)$, and $x^*(t_i)$ is the point in $W^0(t_i)$ which is closest to the point $x(t_i)$. The inequality $\|x(t_i) - x^*(t_i)\| \le \varepsilon_i$ holds.
The relation

$$W^0(t_{i+1})_{\omega(\Delta_i)} \cap \tilde{X}_\psi(t_{i+1}; t_i, x^*(t_i)) \neq \varnothing \tag{3.15}$$

follows from the inclusions $x^*(t_i) \in W^0(t_i)$ and (3.9)
Then, a point

$$x^*(t_{i+1}) = x^*(t_i) + \Delta_i f^*(t_i), \quad f^*(t_i) \in F_\psi(t_i, x^*(t_i)) \tag{3.16}$$

exists which is contained in $W^0(t_{i+1})_{\omega(\Delta_i)}$.
Since $(t_i, x^*(t_i)) \in W^0 \subset \Phi_\sigma$ and $(t_i, x(t_i)) \in W^0_{\varepsilon_i} \subset W^0_\sigma \subset \Phi_\sigma$ then, according to condition A.4, the inequality

$$d(F_\psi(t_i, x(t_i)), F_\psi(t_i, x^*(t_i))) \le \lambda \|x(t_i) - x^*(t_i)\|$$

is satisfied.
Taking this inequality into account, we select a vector $f(t_i) \in F_\psi(t_i, x(t_i))$ which satisfies the inequality

$$\|f(t_i) - f^*(t_i)\| \le \lambda \|x(t_i) - x^*(t_i)\| \le \varepsilon_i$$

It is then found that the point $x(t_{i+1}) = x(t_i) + \Delta_i f(t_i)$ is at a distance no greater than the quantity

$$\|x(t_i) - x^*(t_i)\| + \Delta_i \|f(t_i) - f^*(t_i)\| \le (1 + \lambda\Delta_i)\varepsilon_i$$

from the point (3.15).
This means that $x(t_{i+1}) \in \hat{W}^{(n)}(t_{i+1})$.
Hence it has been shown that the relation

$$\tilde{X}_\psi(t_{i+1}; t_i, x(t_i)) \cap \hat{W}^{(n)}(t_{i+1}) \neq \varnothing$$

is satisfied for any $\psi \in \Psi$ and any $t_i \in T_n$ and $x(t_i) \in \hat{W}^{(n)}(t_i)$,
The inclusion (3.14) follows from this.
The inclusion

$$\hat{W}^{(n)}(t_i) \in \tilde{W}^{(n)}(t_i), \quad t_i \in T_n \tag{3.17}$$

also holds.
We will prove this by mathematical induction. Actually, relations

$$\hat{W}^{(n)}(t_i) = W^0(t_i)_{\varepsilon_i} \subset \Phi(t_i)_{\varepsilon_i}, \quad t_i \in T_n \tag{3.18}$$

$$\hat{W}^{(n)}(t_{N(n)}) = W^0(t_{N(n)})_{\varepsilon_{N(n)}} = M_{\varepsilon_{N(n)}} = \tilde{W}(t_{N(n)}) \tag{3.19}$$

are satisfied. Consequently, the inclusion $\hat{W}^{(n)}(t_i) \subset \hat{W}^{(n)}(t_i)$ is satisfied for $i = N(n)$.
We will now prove that the inclusion (3.17) is satisfied for all remaining i for which $t_i \in T_n$.
In order to do this, we assume that $t_i \in T_n$, $i \le N(n) - 1$ and that the inclusion

$$\hat{W}^{(n)}(t_{i+1}) \subset \tilde{W}^{(n)}(t_{i+1}) \tag{3.20}$$

holds for the instant of time $t_{i+1} \in T_n$.
We will prove that $\hat{W}^{(n)}(t_i) \subset \hat{W}^{(n)}(t_i)$. Actually, it follows from relations (3.14) and (3.18) that

$$\hat{W}^{(n)}(t_i) \subset \Phi(t_i)_{\varepsilon_i} \cap \tilde{X}_\psi^{-1}(t_i; t_{i+1}, \hat{W}^{(n)}(t_{i+1})), \quad \psi \in \Psi$$

and it follows from (3.20) that

$$\Phi(t_i)_{\varepsilon_i} \cap \tilde{X}_\psi^{-1}(t_i; t_{i+1}, \hat{W}^{(n)}(t_{i+1})) \subset \Phi(t_i)_{\varepsilon_i} \cap \tilde{X}_\psi^{-1}(t_i; t_{i+1}, \tilde{W}^{(n)}(t_{i+1})), \quad \psi \in \Psi$$

We therefore obtain that $\hat{W}^{(n)}(t_i) \subset \hat{W}^{(n)}(t_i)$. Relations (3.17) are proved.

We will now use relations (3.17) to prove the inclusion $W^0 \subset \Omega^0$.

When $t_* = \vartheta$, the equalities $W^0(t_*) = M$, $\Omega^0(t_*) = M$ are satisfied and this means that $W^0(t_*) = \Omega^0(t_*)$.

Suppose $t_* < \vartheta$. We choose an arbitrary point and the inequalities

$$t_* \le t_n(t_*) \le t_* + \Delta^{(n)}, \quad n = 1, 2, \dots$$

hold.

Since $(t_*, x_*) \in W^0$, a solution $x(t)$, $t \in [t_*, \vartheta]$ of the differential inclusion $\dot{x} \in F_\psi(t, x), x(t_*) = x_*$ exists for any $\psi \in \Psi$ which satisfies the inclusion $(t, x(t)) \in W^0$, $t \in [t_*, \vartheta]$. The inclusion

$$x(t_n(t_*)) \in W^0(t_n(t_*)) \subset \hat{W}^{(n)}(t_n(t_*)) \subset \tilde{W}^{(n)}(t_n(t_*))$$

follows from this.

This means that a point $x(t_n(t_*)) \in \tilde{W}^{(n)}(t_n(t_*))$ exists for each n such that

$$\|x(t_n(t_*)) - x_*\| \le K(t_n(t_*) - t_*)$$

On taking the equality $\lim(t_n(t_*) - t_*) = 0$ into account, we obtain that the sequence $\{(t_n(t_*)), x(t_n(t_*))\}$ of points from W^0 satisfies the relation $(t_*, x_*) = \lim(t_n(t_*), x(t_n(t_*)))$ and this means that $(t_*, x_*) \in \Omega^0$.

It has been shown that $W^0(t_*) \subset \Omega^0(t_*)$, $t_* < \vartheta$.

The inclusion $W^0 \subset \Omega^0$ follows from the relations $W^0(\vartheta) \subset \Omega^0(\vartheta)$ and $W^0(t_*) \subset \Omega^0(t_*)$, $t_* < \vartheta$. The equality $\Omega^0 = W^0$ follows from the inclusions $\Omega^0 \subset W^0$ and $W^0 \subset \Omega^0$. The theorem is proved.

4. NUMERICAL MODELLING FOR SECOND-ORDER CONTROLLED DYNAMICAL SYSTEMS

A class second-order controlled dynamical systems is considered in which there is no controlling action of the second player. For these systems, the positional absorption set W^0, corresponding to the time interval $[t_0, \vartheta]$, is the set of all the initial positions (t_*, x_*) from which the problem of the adduction of the motions of the system into the target set M after a time not exceeding $\vartheta - t_0$ is solvable.

The problem of the numerical construction of the set W^0 is solved. For this purpose, a subdivision $\Gamma = \{t_0, t_1, \dots, t_n = \vartheta$ of the time interval $[t_0, \vartheta]$ is constructed with a step size Δ. The domain in the phase plane, which "a priori" contains time sections of the set W^0, is covered with a square mesh with a step size h which is proportional to $\Delta^{\frac{3}{2}}$. The union of the sets $\{\tilde{W}(t_i)\}$, which approximates W^0, is constructed on this mesh using the retrograde procedures described above. The discrimination of the boundary of each element of the union is accomplished using the pixel method. A similar approach to the problem of constructing numerical approximations of the solutions of dynamical systems has been used earlier in [16]. The matching of the step size of the time interval subdivision and the step size of the subdivision of the phase space using the formula $h = C\Delta^{3/2}$, where C is a constant, ensures the convergence of the finite-difference constructions to the set W^0 in the Hausdorff metric in the class of problems being considered.

The computational scheme developed was used to model different controllable dynamical system in a plane.

As an example, we consider a controlled rigid spring which is described by the non-linear system of differential equations

$$\dot{x} = y, \quad \dot{y} = -x - x^3 + u; \quad |u| \le 1$$

We will study the evolution of the dynamical system in the time interval $[t_0, \vartheta] = [0, 3]$ when there is a

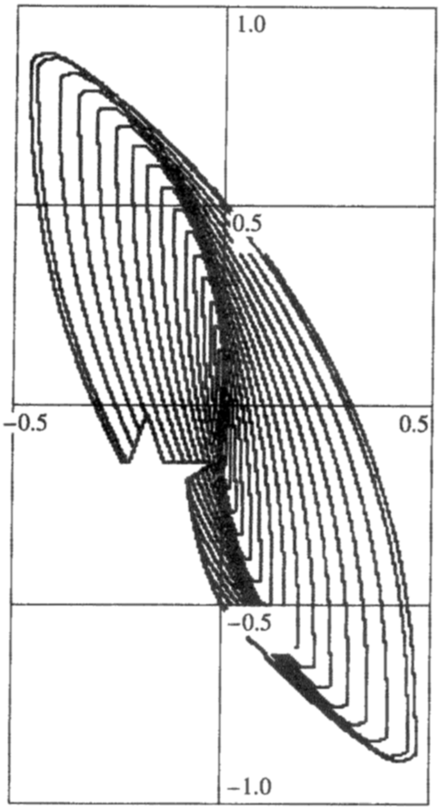

Fig. 1

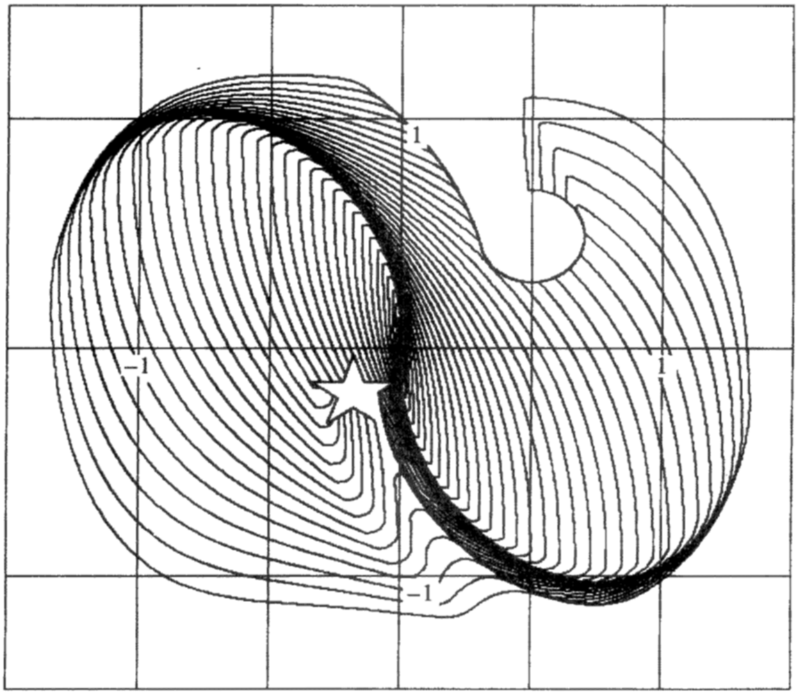

Fig. 2

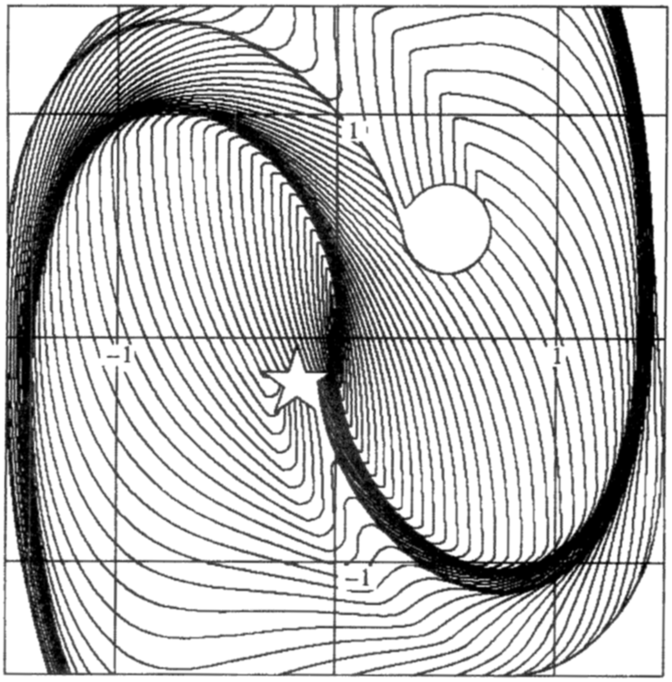

Fig. 3

non-simply-connected phase constraint. As this phase constraint, we consider the sum of a convex set (a circle of radius $r = 0.2$ with its centre at the point $(x, y) = (0.5, 0.5)$, a non-convex set (a five-pointed star with its centre at the point $(x, y) = (-0.13, -0.15)$) and an unbounded set (closure of the complement of the rectangle $B = \{(x, y)$: $-1.5 \leq x \leq 1.5, -1.5 \leq y \leq 1.5$ up to R^2). The origin of coordinates $(x, y) = (0, 0)$ is the objective set. The set W^0 is constructed which can be subsequently used to solve the problem of bringing the motions of the dynamical system to the origin of coordinates.

Sections of the set W^0, corresponding to the instants of time $t = 1, 2, 3$ respectively are shown in Figs 1–3. In the numerical modelling, the discretization parameter $\Delta = 0.02$.

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